

Calculus Memory Book



for use with

AP CALCULUS BC

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Calculus Memory Book

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PRECALCULUS TOPICS

- $a^2 - b^2 = (a+b)(a-b)$
 - Rule of thumb --- Multiply out numerators but keep denominators factored
 - Flip the inequality sign over if \times or \div by a negative
 - $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
 - $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
 - $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
 - If you choose to approximate an answer with a decimal, round to **3** places but **not** till the very end of the problem.
 - Quadratic Formula: Given $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 - To solve any inequality, mark undefined points and solutions to the equality on a number line (split points) and test #'s inbetween the split points in the inequality (test points).
 - $|x| \Rightarrow x$ if $x \geq 0$
 $\Rightarrow -x$ if $x < 0$
 - One way to solve absolute value problems is to split them into separate problems without absolute value
 - Distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
 - Equation of a Circle with center (h,k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$
 - Midpoint formula: $((x_1+x_2)/2 , (y_1+y_2)/2)$
 - Slope of a line: $m = (y_2 - y_1)/(x_2 - x_1)$
 - Point slope formula : $y - y_1 = m(x - x_1)$ (equation of a line)
 - $x=a$, vertical line (infinite or undefined slope)
 - $y=b$, horizontal line (zero slope)
 - Distance from (x_1, y_1) from $Ax + By + C = 0$ is $\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$
 - \Leftrightarrow Parallel lines have same slope
 - \Updownarrow Slopes of perpendicular lines are negative reciprocals
 - y-intercept set $x=0$
 - x-intercept set $y=0$
- The graphs of the following functions should be memorized :
- $y = mx + b$

Notes:

- $y = x^2$
- $y = x^3$
- $y = \sqrt{x}$
- $y = 1/x$
- $y = \sqrt[3]{x}$
- $y = \ln x$
- $y = e^x$
- $y = \sin x$
- $y = \cos x$
- $y = \tan x$

- You need to memorize the unit circle
- You need to memorize the following identities:

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x} \quad \sec x = 1 / \cos x \quad \csc x = 1 / \sin x$$

$$\sin^2 x = 1 - \cos^2 x \quad \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x \quad \tan^2 x = \sec^2 x - 1$$

$$2 \sin x \cos x = \sin 2x \quad \cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\sin(-x) = -\sin x \quad \cos(-x) = \cos x \quad \tan(-x) = -\tan(x)$$

- Function:

- For each x-value, there can be only one y-value
- Domain = all possible values for x
- Range = all possible values for y

- Even Function : show $f(-x) = f(x)$ (y-axis symmetry)

- Odd function : show $f(-x) = -f(x)$ (origin symmetry)

- to find out where two functions intersect, set them equal & solve for x
- Greatest Integer Function [x] or [[x]] rounds x down to the nearest integer less than or equal to x

- Composite: $(f \circ g)(x) = f(g(x))$

- In calculus we change all logarithms to base e (the natural logarithm)

$$\text{using: } \log_a x = \frac{\ln x}{\ln a}$$

- Laws of logarithms

$$\ln(ab) = \ln a + \ln b \quad \ln(a/b) = \ln a - \ln b \quad \ln(1/a) = -\ln a \quad \ln(a^n) = n \ln a$$

LIMITS

- The limit is the y-value you are getting close to not necessarily the function value itself.
- Definition of Limit:**
Given any $\epsilon > 0$ if there is a corresponding number, $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x)-L| < \epsilon$ then $\lim_{x \rightarrow a} f(x) = L$
- $\lim_{x \rightarrow 0} (\sin x)/x = 1$
- $\lim_{x \rightarrow 0} (1 - \cos x)/x = 0$
- $\lim_{x \rightarrow 0} (1 + c/x)^x = e^c$
- $\lim_{x \rightarrow \infty} (1 + cx)^{1/x} = e^c$
- compound interest: $A(t) = A_0 (1 + r/n)^{nt}$
 n = number of times per year compounding takes place t = # of years
 r = annual interest rate and initial investment = A_0
- continuously compounded = $A_0 e^{rt}$
- As long as each part of a limit problem is defined as a real #, it's ok to evaluate limits piecemeal. (i.e. no neg's under $\sqrt{\quad}$, no 0's in denom.)
- Squeeze Theorem:**
if $f(x) \leq g(x) \leq h(x) \rightarrow L$ and $h(x) \rightarrow L$ then $g(x) \rightarrow L$ also.
- Definition of Continuity:** as $x \rightarrow a$ $\lim_{x \rightarrow a} f(x) = f(a)$
- Intermediate Value Theorem:** if $f(x)$ is continuous and p is between $f(a)$ and $f(b)$, then there is at least one x -value c between a and b such that $f(c) = p$.
- L'Hopital (or L'Hospital)'s Rule** if $g(x)$ and $f(x)$ both $\rightarrow 0$ or both $\rightarrow \pm\infty$ then the limit of $\frac{f(x)}{g(x)}$ equals the limit of $\frac{f'(x)}{g'(x)}$
- Indeterminate forms:** $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, \infty^0, 1^\infty, 0^0$
must be rewritten as quotients before applying L'Hopital's Rule.

DERIVATIVES

- Slope of the secant line through $(a, f(a))$ and $(b, f(b))$ equals Average rate of change in f on $[a, b]$: $[f(b)-f(a)]/(b-a)$
- Slope of tangent at $x=c$
 - $M_{\text{tan}} = \lim_{h \rightarrow 0} m_{\text{sec}} = \lim_{h \rightarrow 0} [f(c+h)-f(c)]/h$ as $h \rightarrow 0$
 - or as $x \rightarrow c$, $\lim_{x \rightarrow c} [f(x)-f(c)]/(x-c)$

Maclaurin Series $f(x) \approx \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

Taylor Series $f(x) \approx \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n$

These approximations are only valid if the remainder $r_n(x)$ approaches zero.
La Grange Remainder Theorem

$$F(x) = \sum_{k=0}^n \frac{f^k(a)}{k!} (x-a)^k + r_n(x)$$

where $r_n(x) = \frac{f^{n+1}(c)}{(n+1)!} (x-a)^{n+1}$

where c is between x and a

Note: if the series alternates then the remainder is $<$ the next term and La Grange is NOT needed

Famous Maclaurin Series

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$
- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ only if $-1 < x < 1$
- $\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + x^4 + \dots$ only if $-1 < x < 1$
- $\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + x^4 - \dots$ only if $-1 < x < 1$

You need to know everything in this book backwards and forwards.
You need to practice problems until you can do them in your sleep.

- Divergent series' very nature can be altered if terms are rearranged... therefore you may NOT rearrange them safely

Integral Test

- $\int_a^{\infty} f(x) dx$ converges if and only if $\sum a_n$ converges
 - $\int_a^{\infty} f(x) dx$ diverges if and only if $\sum a_n$ diverges
- p - series**
- $\sum \frac{1}{n^p}$ converges for $p > 1$, diverges for $p < 1$
 - Harmonic series ($p=1$) $\sum \frac{1}{n}$ diverges

Comparison Test (for a_n and $b_n \geq 0$)

- If $a_n \geq b_n$ and $\sum b_n$ diverges then so does $\sum a_n$
- If $a_n \leq b_n$ and $\sum b_n$ converges then so does $\sum a_n$

Limit Comparison Test ($a_n \geq 0$)

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \in \mathbb{R}^+$ (not zero) then either

$\sum a_n$ and $\sum b_n$ both diverge or both converge

Ratio Test ($a_n \geq 0$)

- If $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = p$ then
- If $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = p$ then
 - $p > 1 \rightarrow \sum a_n$ diverges
 - $p < 1 \rightarrow \sum a_n$ converges
 - $p = 1 \rightarrow$ test fails \rightarrow try comparison test

Alternating Series Test

If $\lim_{n \rightarrow \infty} a_n = 0$, $|a_n|$ decreasing and $\sum a_n$ alternates then $\sum a_n$ converges
 If $\sum a_n$ converges but $\sum |a_n|$ diverges then the convergence of $\sum a_n$ is "conditional"
 If both $\sum a_n$ and $\sum |a_n|$ converge then the convergence of $\sum a_n$ is "absolute"

Power Series

$\sum C_n x^n$ will either:

- converge only for $x = 0$
- converge on $(-R, R)$ for some real number R
- converge for all x

(find this interval of convergence by applying the ratio or root test to $|C_n x^n|$)

- These are all the same idea: Slope of tangent line, instantaneous rate of change, slope of curve, derivative
- derivative of $f(x)$ at $x=a$
 $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$
- General formula for a derivative (or slope) of f at any x
 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$
 $f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y / \Delta x = f(x + \Delta x) - f(x)}{\Delta x}$
- If $f'(c)$ exists, then f is continuous at $x=c$
- If f is continuous at $x=c$ and not a sharp corner (cusp or singularity – no local linearity) nor vertical, then $f'(c)$ exists
- If $f'(c)$ does not exist, then f is either discontinuous, has a sharp corner at $x=c$, or has a vertical tangent line at $x=c$.

- $d/dx (c) = 0$
- $d/dx (mx+b) = m$
- $d/dx (x^n) = n * x^{n-1}$ (variable to a constant power)
- $d/dx (k * f(x)) = k * f'(x)$ (Constant Multiplier Rule)
- $d/dx (f+g) = f' + g'$
- Product Rule: $d/dx [f * g] = f * g' + g * f'$
- Triple Product Rule: $d/dx [f * g * h] = f * g * h' + f * g' * h + f' * g * h$
- Quotient Rule: $[g * f' - g' * f] / (g^2) = d/dx (f/g)$
- $d/dx (\sin x) = \cos x$
- $d/dx (\cos x) = -\sin x$
- $d/dx (\tan x) = \sec^2 x$
- $d/dx (\sec x) = \tan x \sec x$
- $d/dx (\csc x) = -\csc x \cot x$
- $d/dx (\cot x) = -\csc^2 x$
- $d/dx (e^x) = e^x$
- $d/dx (a^x) = (\ln a)(a^x)$
- $d/dx (e^{f(x)}) = e^{f(x)} f'(x)$
- $d/dx (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $d/dx (\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $d/dx (\tan^{-1} x) = \frac{1}{1+x^2}$ $d/dx (\sec^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}$
- $d/dx (\ln x) = d/dx (\ln|x|) = d/dx (\ln(-x)) = 1/x$
- $d/dx (\ln(f(x))) = d/dx (\ln|x|) = \frac{f'(x)}{f(x)}$

- $d/dx(|x|) = \frac{|x|}{x}$
- **Chain Rule** (for composites): $d/dx [f(g(x))] = f'(g(x)) * g'(x)$ (tag or baby)
- $dy \leftarrow y$ is the dependent variable
 $dx \leftarrow x$ is the independent variable
 (derivative of y with respect to x)
- **Chain Rule** (Liebniz notation) $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$
- $\Delta y/\Delta x$ = slope of secant line, dy/dx = slope of tangent line
- $x(t)$ = position
- $x'(t) = v(t)$ = velocity
- $x''(t) = a(t)$ = acceleration
- particle/body is at rest if $v(t) = 0$
- particle/body moves left/down if $v(t) < 0$
- particle/body moves right/up if $v(t) > 0$
- speed = $|v(t)|$
- if $a(t)$ and $v(t)$ have same sign then particle is speeding up
- if $a(t)$ and $v(t)$ have opposite signs then particle is slowing down
- Normal line is perpendicular to the tangent line at the point of tangency.
 The slope of the normal line through the point of tangency at $x = a$ is:
 $m = -1 / f'(a)$
- d/dx (variable ^{variable}) = ?
 1. Write $y =$ variable ^{variable}
 2. Take the natural log (ln) of both sides & bring down power
 3. Then use implicit differentiation.
 Inverse Function Theorem
 given (a,b) on $f(x) \Rightarrow (f^{-1})'(b) = 1 / f'(a)$
- If $f(x)$ changes proportionally (or directly) with its own y -value then
 $dy/dx = ky$ and $f(x) = A_0 e^{kx}$
- In a related rates problem "t" is the independent variable.
 All other variables must be multiplied by the appropriate tag/baby.
 ($dV/dt, dy/dt, dA/dt, etc...$)

- $x = r \cos \theta$
- $y = r \sin \theta$
- $x^2 + y^2 = r^2$
- $\tan \theta = y/x$
- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta \sin\theta + r \cos\theta}{dr/d\theta \cos\theta - r \sin\theta}$
- $A = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$
- $\mathcal{L} = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2} d\theta$
- **Vector Valued Functions** : $F = x\vec{i} + y\vec{j} + z\vec{k}$ or $\langle x, y, z \rangle$
 Where x, y, z are functions of t
- $\vec{F}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$
- $\int_{\alpha}^{\beta} \vec{F}(t) dt = \left[\int_{\alpha}^{\beta} x(t) dt \right] \vec{i} + \left[\int_{\alpha}^{\beta} y(t) dt \right] \vec{j} + \left[\int_{\alpha}^{\beta} z(t) dt \right] \vec{k}$
- $\lim_{t \rightarrow a} \vec{F}(t) = \left[\lim_{t \rightarrow a} x(t) \right] \vec{i} + \left[\lim_{t \rightarrow a} y(t) \right] \vec{j} + \left[\lim_{t \rightarrow a} z(t) \right] \vec{k}$
- Smooth curve = no discontinuities and no sharp corners, and $\vec{F}'(t)$ exists and is non zero $\vec{0}$ for all t
- Vector valued functions are really the same as parametric curves.
 They just use different notation.

SEQUENCES AND SERIES

Sequences

- Converge to L if $\lim_{n \rightarrow \infty} a_n = L =$ a real number (diverges otherwise)
- Bounded monotonic sequences converge.
 When in doubt look at a_{100} or a_{1000} etc. – but watch for calculator error if you make n too big for it to handle
- Geometric Series:**
 $a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$ only if $-1 < r < 1$

General Series Rules:

- If a_n does not $\rightarrow 0$ then $\sum a_n$ diverges
- If $a_n \rightarrow 0$ then $\sum a_n$ **might** converge (Has to go to 0 "quickly")
- Convergent series with all positive terms will still converge to the same # even when terms are rearranged.

SLOPE FIELDS

- Euler's Method

Create a chart using the following values: $\Delta x, n, x_n, y_n \approx y_{n-1} + \Delta x \cdot y'(x_{n-1}, y_{n-1})$

X	$\Delta y = y'(previous) \cdot \Delta x$	New y
x_0	-----	y_0
$x_0 + \Delta x$	$y'(x_0, y_0) \cdot \Delta x$	$y_0 + \Delta y$
...
x_n	$y'(x_n, y_n) \cdot \Delta x$	y_n

Euler's method follows pieces of approximate tangent lines.

These values will be a bit low if f is concave up and a bit too large if f is concave down

Slope fields show lots of little pieces (line segments) of the tangent lines for many possible curves with various possible initial conditions

- Given a differential equation:
 - separate the variables: $f(y) dy = g(x) dx$
 - integrate both sides (+c only on one side)
 - solve for c
 - solve for y

Parametric and Polar Functions

- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} \times \frac{dt}{dx}$
- $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$
- $\mathcal{L} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
- $\mathcal{S} = \int_a^b 2\pi r y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ OR $\int_a^b 2\pi x y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
↑ about x-axis ↑ about y-axis
- Speed = $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ (derivative of integral for arc length...duh!)

- In a related rates problem, never substitute values for any variable quantities until after you take the derivative with regards to "t". (Otherwise your derivatives will be zero and answers invalid.)
- Given the tangent line $y = mx + b$ tangent to graph at x
 $y(x + \Delta x) \approx m(x + \Delta x) + b$
 $\approx mx + m\Delta x + b$
 $\approx f(x) + f'(x) \cdot \Delta x$ where $f'(x) \cdot \Delta x = dy \approx \Delta y$
- $f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x$ or $f(x+h) \approx f(x) + f'(x) \cdot h$
 By itself $f'(x) \cdot \Delta x$ is called the "error" or approximate change in f
 The tangent line at (c;f(c)) is: $y = f(c) + f'(c)(x-c)$ note: $x-c = \Delta x$
 So just use the tangent line for linear approximations (linearization)
- Mins and Maxes (extrema) can only occur at *critical points* which are:
 - At an endpoint
 - At stationary points $f'(c) = 0$
 - At singular points $f'(c)$ is undefined but $f(c)$ is defined
- Stationary points are when $f'(c) = 0$, these are extrema
 IF AND ONLY IF f' changes signs at $x = c$
- Inflection points occur where $f'' = 0$ (or undef.) **and** f' changes signs
- Horizontal inflection points occur when $f' = 0$ and $f'' = 0$ and f'' changes signs but f' does not
- Monotonicity Theorem:
 - If $f' > 0$ on an interval then f is increasing in the interval
 - If $f' < 0$ on an interval then f is decreasing in the interval
 - If $f' = 0$ on an interval then f is constant on the interval
- Concavity
 - If $f'' > 0$ then f is concave up on the interval
 - If $f'' < 0$ then f is concave down on the interval
 - f can only change concavity where $f'' = 0$ or is undefined.
 - f'' must also change signs in order for there to actually be an inflection point
- A local (or relative) Max is a y-value greater than neighboring y-values
- A local (or relative) Min is a y-value less than neighboring y-values
- A global (or absolute) Max is the *highest* y-value of all y values in the range
- A global (or absolute) Min is the *lowest* y-value of all y values
- A closed interval guarantees global extrema
- First Derivative Test:** (always works) Given $f'(c) = 0$ or undefined (with $f(c)$ defined) then:
 - If f' changes from $+$ to $-$, $f(c)$ is a maximum
 - If f' changes from $-$ to $+$, $f(c)$ is a minimum
 - The y-value is the minimum or maximum **value**
 - The x-value is called the **critical** value or number

• **Second Derivative Test:**

- Given $f'(c) = 0$ then:
- o If $f'' > 0$ then $f(c)$ is a minimum ☺
 - o If $f'' < 0$ then $f(c)$ is a maximum ☹
 - o If $f'' = 0$ or undefined then the second derivative test fails and you must use first derivative test
- **Never Forget!!** The absolute Min or Max could occur at an endpoint

• **ECONOMICS:**

- $P(x) = R(x) - C(x)$ = profit (profit = revenue - cost)
- $p(x)$ = price
- x = number of units produced/consumed
- $C(x)$ = fixed cost + cost per item * x
- $R(x) = x * \text{price of one item}$
- **Marginal Revenue** = $R'(x)$ **Marginal profit** = $P'(x)$ **Marginal cost** = $C'(x)$

• Vertical Asymptote (can only occur if denominator $\rightarrow 0$, when $x \rightarrow a$)

- o $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty \Rightarrow$ Vertical Asymptote Verification
- Horizontal Asymptote: $y = a$ (to verify...you must take the limit)
- o $\lim_{x \rightarrow \pm\infty} f(x) = a \Rightarrow$ Horizontal asymptote
 - o Note : if $f(x)$ is a rational function
 - o $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} (\text{ratio of leading terms})$

• **Mean Value Theorem** (for derivatives)

Given $f(x)$ continuous on $[a, b]$ and differentiable on (a, b) then $f'(c) = [f(b) - f(a)] / (b - a)$ for at least one value of "c" between a and b.

- Instantaneous rate of change in f at $x=c$ is $f'(c)$
- Average rate of change on interval $[a, b]$ is $(f(b) - f(a)) / (b - a)$
- If $f' = g$ then $f - g = c$ (or $f = g + c$)

• **Newton's Method** (to \approx a zero of $f(x)$)

- $x_1 = \#$ of your choice near the zero (1st approximation)
- $x_2 = x_1 - f(x_1) / f'(x_1)$ (2nd approximation)
- $x_3 = x_2 - f(x_2) / f'(x_2)$ (3rd approximation)
- continue until answers repeat

• Rotate about y-axis:

$$\int_c^d \pi [f(y)]^2 dy \quad (\text{disks})$$

$$\pi \int_c^d \{ [f(y)]^2 - [g(y)]^2 \} dy \quad (\text{washers})$$

• Rotate about $y=k$

$$\int_a^b \pi [k - f(x)]^2 dx \quad (\text{disks})$$

$$\pi \int_a^b \{ [k - f(x)]^2 - [k - g(x)]^2 \} dx \quad (\text{washers})$$

• Rotate about $x=k$

$$\int_c^d \pi [k - f(y)]^2 dy \quad (\text{disks})$$

$$\pi \int_c^d \{ [k - f(y)]^2 - [k - g(y)]^2 \} dy \quad (\text{washers})$$

• Arc length

$$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2} dx \quad \text{or} \quad \int_c^d \sqrt{1 + [f'(y)]^2} dy$$

• Speed along the curve = the derivative of the above integral i.e. $\sqrt{1 + [f'(x)]^2}$ etc.

• Surface Area when curve is rotated ...

about x-axis: $S = 2\pi \int_a^b y \sqrt{1 + [f'(x)]^2} dx$ or $2\pi \int_c^d y \sqrt{1 + [f'(y)]^2} dy$

about y-axis: $S = 2\pi \int_a^b x \sqrt{1 + [f'(x)]^2} dx$ or $2\pi \int_c^d x \sqrt{1 + [f'(y)]^2} dy$

• **WORK = the integral of force**

$$W = \int_a^b f(x) dx \quad \text{For springs } F(x) = kx$$

For pumping a liquid $W = \int_a^b (d * a * h) dx$

- Where $d = 9.8$ * mass per unit (metric/meters) or weight per unit (standard/pounds) or density
- $a =$ area of cross section of liquid to be moved
- $h =$ distance each slab must travel
- $dx =$ the instantaneous thickness of each cross section and a to b is original position of liquid
- Work units are Joules or foot-pounds
- Force units are Newtons or pounds
- Note: average speed = total distance traveled / total time
- average velocity = displacement / total time

INTEGRATION

- **integration by partial fractions:**

$$\int \frac{f(x)}{(x-a)(x-b)\dots(x-c)} dx = \int \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{C}{x-c} dx$$

repeated powers need extra fractions. (ex $\frac{1}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$)

If the degree of the numerator is greater than or equal to the degree of the denominator, then the fraction is "top heavy" and long division must be applied **BEFORE** using the technique of partial fractions.

- **integration of trigonometric integrals:**

odd powers of sine and cosine – save one, convert rest using

Pythagorean identities

even powers of sine and cosine – use $\frac{1}{2}(1 \pm \cos 2x)$ repeatedly

- **integration by desperation** – try some creative u-substitutions. Try letting u = entire radical and remove the radical before computing du.

APPLICATIONS OF INTEGRALS

- Area between two curves that intersect at x=a and x=b

$$= \int_a^b |f(x) - g(x)| dx$$

(avoid absolute value by placing higher function first)

- Total Distance Traveled

$$= \int_a^b |v(t)| dt + \int_c^d -|v(t)| dt \quad \text{OR} \quad \int_a^d |v(t)| dt$$

(where v(t) > 0) (where v(t) < 0) (total distance moved)

total dist. moving right total dist. moving left

- Displacement

$$\int_a^b v(t) dt \quad \text{where } t=a \text{ is starting time}$$

where t=b is ending time

(distance between starting position and ending position)

VOLUME BY DISKS & WASHERS

- Rotate about x-axis

$$\int_a^b \pi [f(x)]^2 dx \quad \text{(disks)}$$

$$\pi \int_a^b \{ [f(x)]^2 - [g(x)]^2 \} dx \quad \text{(washers)}$$

Be sure that you **square the radii separately** and then subtract!

- $\int x^r dx = \frac{1}{r+1} x^{r+1} + c$
- $\int \sin x dx = -\cos x + c$
- $\int \cos x dx = \sin x + c$
- $\int k f(x) dx = k \int f(x) dx$
- $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$
- $\int [g(x)]^r g'(x) dx = \frac{1}{r+1} [g(x)]^{r+1} + c$
- $\int \sec^2 x dx = \tan x + c$
- $\int \sec x \tan x dx = \sec x + c$
- $\int -\csc^2 x dx = \cot x + c$
- $\int \csc x \cot x dx = -\csc x + c$
- $\int f'(g(x)) g'(x) dx = f(g(x)) + c$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \frac{1}{x} dx = \ln |x| + C$
- $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

- **Trapezoidal Rule** – using trapezoids to approximate area, bases are y-values, heights are Δx and each trapezoid's area = $\frac{1}{2} h (b_1 + b_2)$
 $A = \frac{1}{2} \Delta x [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$
 (Double all but first and last y-values.)

- **Fundamental Theorem of Calculus:**

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F(x) \text{ is any antiderivative of } f(x)$$

i.e. $F'(x) = f(x)$

- Version two of the Fundamental Theorem: $d/dx [\int_a^x f(t) dt] = f(x)$
- If $f(x) \geq 0$, you get the area
- If $f(x) \leq 0$, you get the negative of the area
- any Lower Sum $\leq \int_a^b f(x) dx \leq$ any Upper Sum
- $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$ (upper and lower sums with one rectangle)
 $\uparrow \qquad \qquad \qquad \uparrow$
 min on [a,b] max on [a,b]
- $d/dx [\int_{h(x)}^{g(x)} f(t) dt = f(g(x))g'(x) - f(h(x))h'(x)$

- **Mean Value Theorem for Integrals:**

$$\int_a^b f(t) dt = f(c)[b-a] \quad \text{for some } c \text{ between } a \text{ and } b.$$

Note: $f(c)$ is called the **average value** of $f(x)$ on $[a,b]$.

$$f(c) = \frac{\int_a^b f(x) dx}{b-a} = \frac{F(b) - F(a)}{b-a}$$

(not to be confused with average rate of change in $f(x) = \frac{f(b) - f(a)}{b-a}$)

- **U-substitution:**

If you see an integral with a composite function and a more or less correct tag/baby, then let u equal the inside function, compute du. Change everything totally into u's and integrate. **THIS SHOULD BE FULLY MASTERED** and the FIRST THING YOU TRY!!

Don't forget to change a and b also!

- **integration by parts:** $\int f' \cdot g = fg - \int f \cdot g$ or $\int u dv = uv - \int v du$

- $\int \sec x dx = \ln |\sec x + \tan x| + C$
- $\int \tan x dx = \ln |\sec x| + C = -\ln |\cos x| + C$
- $\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = n(n+1)/2$
- $\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = n(n+1)(2n+1)/6$
- $\sum_{i=1}^n c = nc$

- Riemann Sum – any combination of rectangles whose tops intersect the curve of $f(x)$ and which approximate the area between $f(x)$ and the x-axis, using n rectangles from $x=a$ to $x=b$
- Right sum – use the rightmost x-value on each interval to compute $f(x)$ = height of rectangle
- Left sum – use the leftmost x-value on each interval to compute $f(x)$
- Upper sum – use largest y-value on each interval for the heights
- Lower sum – use smallest y-value on each interval for the heights
- If $f(x) > 0$ then upper sums are circumscribed and lower sums are inscribed

- If $f(x) < 0$ then upper sums are inscribed and lower sums are circumscribed
- Midpoint sum – use the x-value in the center of each interval

- $\int_a^a f(x) dx = 0$

- $\int_a^b f(x) dx = -\int_b^a f(x) dx$

- $\int_a^b f(x) dx = \lim_{n \rightarrow \infty}$ (any Riemann Sum with n rectangles)

- i.e. $\sum f(x_i) \Delta x_i$ approaches the integral $\int_a^b f(x) dx$

- $f(x_i)$ = height of each rectangle, Δx = width of each rectangle = $\frac{b-a}{n}$,

and $x_i = a + i \cdot \frac{b-a}{n}$

- $f(x_i) \Rightarrow f(x)$ and $\Delta x \Rightarrow dx$